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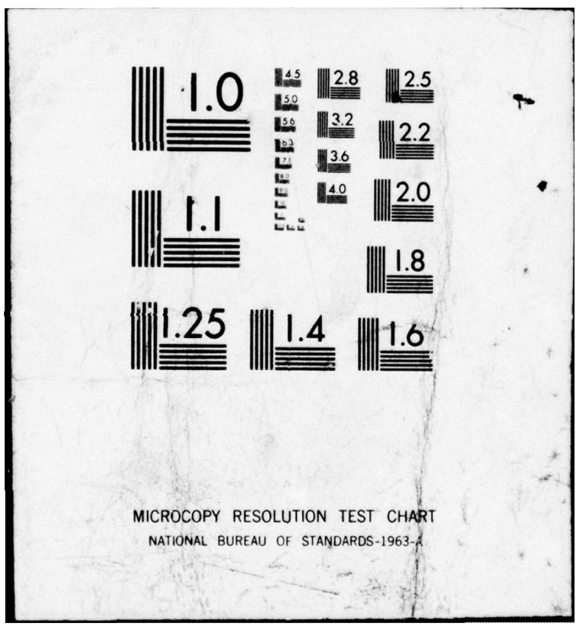
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SOME BOUNDS ON THE GENERALIZED FIRE CONTROL PROBLEM

Harry L. Reed, Jr.

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USA BALLISTIC RESEARCH LABORATORIES
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The general problem of target maneuvers is attacked from the point of view of considering what would be optimal maneuver capabilities for the target given that the attacking gunner can conduct optimal fire control. It is shown that if p is the minimum with respect to target maneuver statistics of the maximum with respect to fire control algorithms of the kill probability, then $p_1 < p < \frac{1}{2}p_1$, where c can be calculated from the correlation function of the maneuver process. Further, if there is a bound on the Mth derivative of the

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target path, $\hat{p}_1$ can be calculated from only the vulnerable area, the time of flight of the projectile, and the r.m.s. value of the Mth derivative.

The results are perfectly general and do not depend on any assumption on the nature of the statistics such as a Gaussian assumption. The results are also simple and should allow a quick analysis when some sort of bounding estimate is required.

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1. INTRODUCTION

This paper started as a tutorial exercise to help tighten up some of the concepts that a number of groups have been referring to somewhat loosely. The idea was to start with a quite general statement of the fire control problem as in equation 2.1 and to go from there to define the roles of dither, noise, and weapon precision, to define the concept of optimal algorithms, and to discuss the idea of optimal maneuvers.

All that is contained here, but in addition there are some more quantitative results as in equation 6.12 which developed as the paper was being written. These allow bounds to be put on the ability of the target to minimize the best that the fire control can do, and they allow this characterization to be expressed by quantities calculated from only the correlation function of the process -- whatever the process -- Gaussian or otherwise.

2. STATEMENT OF THE PROBLEM

The mathematical problem analyzed in this paper is posed in terms that relate it to fire control problems for gun systems. The gunner has data on the behavior of the target over some subset of times taken from the set of times less than or equal to t_o. These data are most likely corrupted by noise. At time t_o the gunner fires. The bullet is committed to its trajectory at time t_o and arrives at the target plane at time $t_o + T$. The target plane can be taken as the plane containing the bullet and a reference point on the target and perpendicular to the relative velocity vector of the bullet and target.

It is usually convenient to consider the distribution of bullet positions in the target plane, and this gives the problem an essentially two spatial dimensional character -- at least if the definition of the target plane can be made unambiguously. Since the prime motivation for this paper was the air defense gun problem, we shall emphasize the two-dimensional problem -- when the dimensionality is important to the discussion. However, we should point out that there is an important class of problems that are fundamentally one-dimensional -- for example, the case in which a vehicle is constrained to move on the earth and that vehicle is being engaged by a gun whose trajectory is virtually parallel to the earth.

The gunner may well fire more than one round at the target. In our analysis we assume that he does not use data on previous hits and misses to correct his fire -- each shot is handled as an independent operation on the part of the gunner.

Likewise, the pilot of the target performs maneuvers that are constrained by physical restrictions on his vehicle, requirements of his mission, and his desire to avoid being hit. The pilot may institute a series of evasive maneuvers when he observes the firing of the gun, but he does not cue the individual twists and turns of his maneuver on the firing of each shot from the gun.

We assume that the gunner has complete knowledge of the statistics of the target's maneuvers. We also assume that the pilot is aware of this fact and also knows that the gunner is smart enough to know how to optimize his fire control on the basis of this statistical knowledge.

In symbols, the problem becomes one of minimizing p (by the pilot) and maximizing p (by the gunner) where

$$p = \int b[q - c - f(z)]j(c)v(q - x)g(x,z)dqdcdxdm(z), \qquad 2.1$$

and where

p = the probability of killing the target given a shot at time t_o

$b(\cdot)$ = the probability density function of the ballistic dispersion of the gun about the aim point

q = the vector position of the bullet in the target plane

c = that extra dispersion called dither that is sometimes added to the aiming of the gun

$f(\cdot)$ = the fire control algorithm. This is a functional relating the past performance of the target to a prediction of the future position of the target.

z = y + n

y = history of the path of the target taken over some subset of the set of times less than or equal to t_o. For example, this may be the history of the target over the interval $t_o \leq t \leq t$. Or the history could be taken over several disjoint intervals as the target pops in and out of view.

n = the noise that represents errors in measuring y. This is, of course, taken over the same subset of times as is y.

$j(\cdot)$ = the probability density function of dither

$v(\cdot)$ = the probability that a hit at the point described by its argument will produce a kill. Note that this is not a probability density function.

x = the future position of the target; that is, the position at time $t_o + T$

$g(\cdot,\cdot)$ = the conditional probability density function of x given z

m(z) = the probability measure over the ensemble Z

Z = the ensemble of functions z

We will also have use for the following:

$m(y)$ = the probability measure over the ensemble Y

Y = the ensemble of possible target path histories y

$m(n)$ = the probability measure over the ensemble N

N = the ensemble of noise histories n

It is more convenient to use $g(x,z)dx$ than the more complete form $d_xG(x,z)$, and we do this also since later results indicate that it is important to the pilot to minimize $\sup_x g(x,z)$.

Usually the extent of the target as represented by $\nu(\cdot)$ is small compared to errors in predicting the future position of the target -- at least in those cases in which those errors are large enough to justify the efforts of this paper. We thus replace $\nu(\cdot)$ by $A\,\delta(\cdot)$ where A is the so-called vulnerable area.

This gives us

$$p = A \int b[x - c - f(z)]j(c)g(x,z)dcdxdm(z). \qquad 2.2$$

We shall use equation 2.2 in all that follows.

3. THE OPTIMAL ALGORITHM

If everything except $f(\cdot)$ is given in equation 2.2, then the gunner need only solve the problem of minimizing p by choosing $f(\cdot)$, and we can write equation 2.2 as

$$p = \int h[f(z),z]dm(z) \qquad 3.1$$

by formally integrating with respect to all the other dummy variables.

We define a set of functionals F which are those $f(\cdot)$ such that $f(z)$ is measurable in Z. Let Z^* be a subset of Z such that the measure of Z^* is unity. Equality over Z^* is the usual equality "almost everywhere."

We assume that for each element $z \in Z^*$ there is an x called $\hat{f}(z)$ such that

$$h[\hat{f}(z),z] = \sup_x h(x,z) \qquad 3.2$$

and we assume that $\hat{f}(\cdot) \in F$. Then

$$\hat{p} = \int h[\hat{f}(z),z]dm(z) \geq \int h[f(z),z]dm(z) \qquad 3.3$$

for all $f(\cdot) \in F$.

The functional $\hat{f}(\cdot)$ is the (perhaps not unique) optimal fire control algorithm. By using it, the gunner takes full advantage of the information available to him. He can do no better. The remainder of his error is in the category of completely unpredictable.

Throughout this paper we shall assume that the gunner should maximize the individual kill probability for each round he fires. This assumption is not strictly true for rounds fired in "bursts." It is a satisfactory assumption for rounds fired at times that are sufficiently different that the target's locations at the two times are not strongly correlated. It is also valid if the individual rounds have a low probability of killing the target given a hit. However, the problem with the assumption is not associated with predicting the target's future position but rather is one of "overkilling" the target. That is, if the gunner has a very effective round, he would prefer to fire a second round at a somewhat different place than his first round -- if the first round hit the target, the target was killed; if the first round missed, why send a second round there?

We shall say more about this issue after we have some of the arguments in the ne section and have established the idea of the relative merit of determ istic algorithms. Nevertheless, we must point out that for gunfire against aircraft we are perhaps concerned about not overlapping a few square meters of vulnerable area while irreducible errors in prediction may spread the target over hundreds of square meters. Also we should point out that estimating the kill of a burst of I rounds as

$$p_I = 1 - (1 - p)^I \qquad 3.4$$

is an overestimate and as such is consistent with the strategy of our analysis in this paper.

4. THE ROLES OF DITHER AND GUN ACCURACY

In this section and in the next, we shall "peel the onion" from the point of view of the gunner. That is, we shall simplify equation 2.2 by optimizing those things over which the gunner has control or over which he could have control if he were to choose to improve his weapon system.

As mentioned above, we shall do this analysis with the assumption that the optimal strategy for the gunner is to maximize p for each round. This is probably an adequate assumption for our purposes, but for completeness at least we shall then show that the ideas of this section can be generalized if this assumption is inadequate.

First consider dither. We shall show that dither is less effective than using the actual target statistics. To do this, we let

$$\hat{p} = A \int b[x - \hat{f}(z)]g(x,z)dxdm(z), \qquad 4.1$$

where $\hat{f}(\cdot)$ is optimal for the ditherless case. That is, let

$$A \int b[x - \hat{f}(z)]g(x,z)dxdm(z) \geq A \int b[x - f(z)]g(x,z)dxdm(z) \qquad 4.2$$

for all $f \in F$.

Then if we let $f(z) + c = f_c(z)$ and note that $f_c(z) \in F$, we have by integrating both sides of equation 4.2 multiplied by $j(c)$

$$A \int \left\{ \int b[x - c - f(z)]g(x,z)dxdm(z) \right\} j(c)\, dc$$

$$\leq \int \{\hat{p}\}\, j(c)dc = \hat{p} \qquad 4.3$$

since $\hat{p}$ is independent of c and $j(\cdot)$ is a probability density function.

Thus "the gunner does at least as well by using the optimal prediction algorithm in the ditherless case as he does by using dither and the same or any other prediction algorithm." Dither has its role, however, if the target's statistics are unknown or for some other reason a less than optimal algorithm is used. This point is illustrated in Appendix A.

For our main effort, however, we shall assume that our "smart" gunner knows $g(\cdot,\cdot)$ and uses $j(\cdot) = \delta(\cdot)$ and optimizes

$$p = A \int b[x - f(z)]g(x,z)dxdm(z). \qquad 4.4$$

We can argue similarly that accurate guns are better than less accurate guns. In particular, if $b(\cdot)$ is the convolution of $b_1(\cdot)$ and $b_2(\cdot)$, that is

$$b(\xi) = \int b_1(\xi - \eta)b_2(\eta)d\eta, \qquad 4.5$$

and if

$$\hat{p}_i = A \int b_i[x - \hat{f}_i(z)]g(x,z)dxdm(z) \qquad 4.6$$

is optimized for each i = 1 and 2, then

$$\hat{p} \leq \hat{p}_i \qquad 4.7$$

for each i where $\hat{p}$ is optimal for

$$p = A \int b[x - f(z)]g(x,z)dxdm(z). \qquad 4.8$$

The proof follows the lines of the proof on dither by substituting equation 4.5 into 4.8 and treating one of the $b_i(\cdot)$ as $b(\cdot)$ and the other as $j(\cdot)$ in equations 4.1, 4.2, and 4.3.

Two special cases are important. First, since

$$b(\xi) = \int b(\xi - \eta)\delta(\eta)d\eta,$$

the gunner does best of all with a perfect gun. So we will take another layer from the onion and henceforth we shall use $b(\cdot) = \delta(\cdot)$ as the best course of action for the gunner. This, by the way, does not represent a physically impossible situation. Guns that are well designed have accuracies that are usually small compared with the other errors of the weapon system -- so much so in fact that additional dispersion (dither) is often added to eliminate biases.

Now we want to optimize (or rather the gunner wishes to optimize)

$$p = A \int g[f(z),z]dm(z) \qquad 4.9$$

which he does by picking $\hat{f}(\cdot)$ so that

$$g[\hat{f}(z),z] \geq g[f(z),z] \qquad 4.10$$

for all $f(\cdot) \in F$ and almost everywhere in z. Then we have

$$\hat{p} = A \int g[\hat{f}(z),z]dm(z). \qquad 4.11$$

Second, since Gaussian functions are usually good approximations for ballistic dispersion, and since if $b(\cdot)$, $b_1(\cdot)$, $b_2(\cdot)$ are Gaussian with standard deviations σ, σ_1, and σ_2, and

$$\sigma^2 = \sigma_1{}^2 + \sigma_2{}^2 \qquad 4.12$$

then

$$b(\xi) = \int b_1(\xi - \eta)b_2(\eta)\ d\eta, \qquad 4.13$$

it follows that a lower dispersion gun (σ_1 or σ_2) is better than a higher dispersion gun (σ) if the optimal prediction algorithm for $b_1(\cdot)$ or $b_2(\cdot)$ is used for the lower dispersion gun and any algorithm is used for the higher dispersion gun.

Now consider a burst of rounds where each round is highly lethal (a hit is a kill) and such that the burst is fired before the first round reaches the target. Certainly the gunner should fire the first round at $x_o = \hat{f}(z)$ as in equation 4.10. We can show by arguments similar to that

above that a second round fired at time t_1 should be aimed at the position x_1 which maximizes $g(x,x_o,z_1)$, the conditional distribution of the target at time $t_1 + T$ given that the target was not at x_o (within the size of the target) at time $t_o + T$ and given the history z_1 up to time t_1.

Likewise a third round fired at time t_2 should be aimed at the position which maximizes $g(x,x_1,x_o,z_2)$, the conditional distribution of positions of the target at time $t_2 + T$ given that the target was not at x_1 at time $t_1 + T$ and was not at x_o at time $t_o + T$ and given the history z_2 up to time t_2. Thus we can define an optimal algorithm which is deterministic. That may be of more theoretical than practical interest -- but there it is.

5. NOISE

As for dither, noise is degrading if the optimal prediction algorithm is used for the noiseless case. To show this, we rewrite equation 4.9 in the form

$$p = A \int g_1[f(y+n),y]dm(y)dm(n), \qquad 5.1$$

where $g_1(\cdot,\cdot)$ is the conditional probability density function of the future position given the actual history of the target's path. The argument of $f(\cdot)$ is, of course, the noisy history. Now let $\hat{f}(\cdot)$ be such that

$$g_1[\hat{f}(y),y] \geq g_1[f(y),y] \qquad 5.2$$

almost everywhere in y. Now, letting $f_n(y) = f(y+n)$, we have $f_n(y) \in F$ and hence

$$g_1[f(y+n),y] \leq g_1[\hat{f}(y),y]. \qquad 5.3$$

Integrating equation 5.3 with respect to dm(y) and dm(n), we get

$$p \leq \hat{p} = A \int g_1[\hat{f}(y),y]dm(y) \qquad 5.4$$

for all $f(\cdot) \in F$.

We shall assume in what follows that the gunner can reduce noise, or will reduce noise, or should not be assumed not to be able to reduce the noise to essentially zero. In any event, we can put an upper bound on his performance by assuming that he is in a noiseless situation.

We have now completely optimized the problem from the point of view of the gunner. The optimal performance from the gunner's point of view is achieved as

$$\hat{p} = \int g[\hat{f}(y),y]dm(y) \qquad 5.5$$

where

$$g[\hat{f}(y),y] \geq g[f(y),y] \qquad 5.6$$

for all $f(\cdot) \in F$,

where

$\hat{p}$ = the best the gunner can do

$g(\cdot,\cdot)$ = the conditional probability density of the future position given the *actual* target history. Note the change in notation.

$\hat{f}(\cdot)$ = the optimal fire control algorithm

y = the actual target history

$m(y)$ = the measure over the ensemble Y of histories y

6. OPTIMAL TARGET MANEUVERS

A reasonable course of action for the pilot is to assume that the gunner is smart and that the thing for the pilot (actually for the ensemble of pilots) to do is to pick $g(\cdot,\cdot)$ and $m(y)$ to minimize $\hat{p}$ in equations 5.5 and 5.6.

Because the gunner can refire, the pilot cannot perform only one maneuver but must carry out a series of maneuvers. For example, if the target had an acceleration limit of a and initiated an arbitrary but fixed acceleration after t_o, the target would be uniformly distributed in a circle of radius $\frac{1}{2}aT^2$ at time $t_o + T$. A little thought shows that this would be optimal for a shot fired at t_o. But if the gunner were then to fire a round at time $t_o + \varepsilon$, he would be able to predict almost exactly where the target (now accelerating constantly) would be at time $t_o + T + \varepsilon$.

This course of action requires some sort of assumption on the dependence of $g(\cdot,\cdot)$ and $m(y)$ on t_o. Typically one would assume that $g(\cdot,\cdot)$ and $m(z)$ were independent of t_o (at least for the evasive components of the motion). The impact of such assumptions on the character of possible maneuvers is discussed in Reference 1. We shall assume this stationarity at least over some time interval of the order of size of the flight time T of the bullet.

In Appendix B we show that if

$$\sigma_r^2 = \int \left\{ [x_1 - \mu_1(y)]^2 + [x_2 - \mu_2(y)]^2 \right\} g(x,y)\,dx\,dm(y) \qquad 6.1$$

where

$$\mu_i(y) = \int x_i g(x,y)\,dx\,dm(y), \qquad 6.2$$

then

$$\hat{p} \geq \frac{A}{2\pi\,\sigma_r^2}. \qquad 6.3$$

[1] S. S. Wolff and P. R. Schlegel, "Fire Control Predictors for Non-Gaussian Target Maneuvers," Ballistic Research Laboratories Report No. 1875, April 1976. (AD #B011320L)

Further, since the second moment is minimized around the mean, we have for any functions $\nu_1(\cdot)$ and $\nu_2(\cdot)$

$$\sigma_r^2 \le \int \left\{ [x_1 - \nu_1(y)]^2 + [x_2 - \nu_2(y)]^2 \right\} g(x,y)dxdm(y) \qquad 6.4$$

In particular, let $\nu_1(\cdot)$ and $\nu_2(\cdot)$ be the optimal (r.m.s.) *linear* estimators of the mean as given by the Wiener theory. If σ_{rw}^2 is the variance from the Wiener theory, then by equation 6.4

$$\sigma_r^2 \le \sigma_{rw}^2 \qquad 6.5$$

and hence

$$\hat{p} \ge \frac{A}{2\pi\sigma_{rw}^2} \cdot \qquad 6.6$$

This is a very useful relationship. σ_r^2 may well not be easy to come by. However, σ_{rw}^2 is obtainable from just the correlation function $\phi(\cdot)$ without any further knowledge of the process (see References 2 and 3). Even though the optimal linear estimator may be a poor estimator of the mean and/or the optimal fire control algorithm, we can still formally calculate σ_{rw}^2 and still use it in equation 6.6.

If, however, the pilots assume a Gaussian ensemble of maneuvers, the Wiener estimator becomes the optimal fire control algorithm, and the distribution of prediction errors is

$$\frac{e^{-\frac{[(\Delta x_1)^2 + (\Delta x_2)^2]}{\sigma_{rw}^2}}}{\pi\sigma_{rw}^2} \qquad 6.7$$

where Δx_1 and Δx_2 are measured from the predicted mean. Since the predicted mean is the aim point, and with our definition of vulnerable area, we have for this case

$$\hat{p} = \frac{A}{\sigma\pi_{rw}^2} \cdot \qquad 6.8$$

So if $\hat{\hat{p}}$ is the best the pilot could do in minimizing p, we have

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[2]Norbert Wiener, *Extrapolation, Interpolation, and Smoothing of Stationary Time Series*, The Technology Press of M.I.T. and John Wiley & Sons, Inc., New York.

[3]H. W. Bode and C. E. Shannon, "A Simplified Derivation of Linear Least Square Smoothing and Prediction Theory," *Proceedings of the Institute of Radio Engineers*, Vol. 38, pp. 417-425, April 1950.

$$\frac{1}{\sigma\pi^2_{rw}} \geq \hat{\hat{p}} \geq \frac{1}{2\sigma\pi^2_{rw}} . \qquad 6.9$$

The left side obtains since the pilot can do that well by using a Gaussian process. How close he can get to the right side of 6.9 is an open question, however (see Reference 1).

Equation 6.9 is especially useful since we can incorporate constraints. For example, in Reference 4 (and outlined in Appendix C) the following problem is solved:

Maximize σ^2_{rw} subject to

$$E\left\{[x_1^{(M)}(t)]^2 + [x_2^{(M)}(t)]^2\right\} = C_M^2. \qquad 6.10$$

The solution is

$$\sigma^2_{rwM} = k_M C_M^2, \qquad 6.11$$

where k_M is the largest eigenvalue of a differential equation.

Now this is as large as the pilot can make σ^2_{rw} subject to an r.m.s. constraint on his Mth derivative. Thus

$$\frac{1}{\pi\sigma^2_{rwM}} \geq \hat{\hat{p}} \geq \frac{1}{2\pi\sigma^2_{rwM}} \qquad 6.12$$

for all processes for which the Mth derivative has the r.m.s. value of C_M.

So "for any type of maneuver process we can bound within a factor of two the *best* that a pilot can do against the best a gunner can do where the vehicle has an r.m.s. constraint on its Mth derivative." How close the pilot can come to achieving the right-hand side of equation 6.12 is still a matter for conjecture.

7. CONCLUSIONS

In addition to the fact that equation 6.12 allows bounds to be put on p, we can draw some other conclusions:

- If the pilot is forced to use maneuver tactics which result in peaks in g(x,z), the fire control designer is well advised to create optimal algorithms and design an accurate and quiet fire control system

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[4] Harry L. Reed, Jr., "A Simple Model of Gunfire Predictors," Ballistic Research Laboratories Report No. 1775, April 1975. (AD #A009507)

to take advantage thereof. The size of the benefits which might accrue in this instance has not been discussed in this paper (except that $\hat{p} \geq \frac{A}{2\pi\sigma_r^2}$), but some idea of possible advantages is given in Reference 5.[5]

- If the pilot has rather complete freedom, he can achieve the performance specified in equation 6.12. He does this with either an ensemble of Gaussian maneuvers or with an ensemble with even a "flatter" (if a heuristic concept can be exercised) conditional probability density.

- Assuming an optimal fire controller, there is a hit probability associated with a Gaussian target tactic; the pilot can do no better than reduce this probability by half.

- A word of caution is appropriate here. If the target is a high-performance aircraft and if the pilot is indeed able to conduct optimal maneuvers, the choice of an optimal fire control algorithm may be academic. That is, the optimal algorithm may be only the least poor of a set of poor choices.

- It is important to know more about the statistics of targets:

  - Will the maneuvers be mild and uniformly distributed?

  - Will there be non-Gaussian eccentricities that offer significant gains to an optimal algorithm?

  - Will the targets maneuver so severely and unpredictably that even optimal algorithms would be inadequate at certain ranges of interest?

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[5] Harry L. Reed, Jr., "Limitations of the R.M.S. Criterion for Fire Control," Ballistic Research Laboratories Report No. 1805, July 1975. (AD #A014986)

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4. Harry L. Reed, Jr., "A Simple Model of Gunfire Predictors," Ballistic Research Laboratories Report No. 1775, April 1975. (AD #A009507)

5. Harry L. Reed, Jr., "Limitations of the R.M.S. Criterion for Fire Control," Ballistic Research Laboratories Report No. 1805, July 1975. (AD #A014986)

APPENDIX A

Herein we shall consider a simple bimodal model that shows the problems one can get into by using the mean as a predictor. The model seems to be almost too simple to be real, but there are at least two real analogues: a target that can maneuver hard left or hard right immediately after a bullet is launched, and two targets which are causing a radar tracking system to "glint" between them.

Anyway, in equation 2.1 let

$$g(x) = \frac{1}{2}[\delta(x-1) + \delta(x+1)]$$

$$\begin{aligned} v(a-x) &= 1 \qquad |a-x| \le \varepsilon \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

$$b(\cdot) = \delta(\cdot)$$

We note that this g is independent of z, so we can take $f(\cdot)$ to be independent of z. Integrating and changing some dummy variables, we have

$$p = \frac{1}{2}\int_{f-1-\varepsilon}^{f-1+\varepsilon} j(c)dc + \frac{1}{2}\int_{f+1-\varepsilon}^{f+1+\varepsilon} j(c)dc$$

and if $0 < \varepsilon < 1$, the regions of integration do not overlap, so

$$p < \frac{1}{2}\int_{-\infty}^{\infty} j(c)dc = \frac{1}{2}\,.$$

So for any prediction algorithm and any dither, p is bounded by $\frac{1}{2}$.

An optimum algorithm would be one for which p equals $\frac{1}{2}$. A class of optimum algorithms is:

$$\begin{aligned} \text{Let}\quad f &= 1 \text{ with probability } \alpha \\ &= -1 \text{ with probability } 1-\alpha. \end{aligned}$$

Then we would have

$$p = \alpha \left\{ \frac{1}{2} \int_{-\varepsilon}^{\varepsilon} j(c)dc + \frac{1}{2} \int_{2-\varepsilon}^{2+\varepsilon} j(c)dc \right\}$$

$$+ (1-\alpha) \left\{ \frac{1}{2} \int_{-2-\varepsilon}^{-2+\varepsilon} j(c)dc + \frac{1}{2} \int_{-\varepsilon}^{\varepsilon} j(c)dc \right\}$$

We can take

$$j(\cdot) = \delta(\cdot)$$

and have

$$p = \frac{1}{2}\alpha + \frac{1}{2}(1 - \alpha) = \frac{1}{2}.$$

Note that this can be a completely deterministic algorithm by choosing $\alpha = 0$ or $\alpha = 1$. Clearly as ε goes down in size $j(c)$ must concentrate its probability mass at 0, 2, or -2 if we are to achieve $p = 1/2$.

If, on the other hand, we were to use the mean as the predictor, we would set $f = 0$ and have

$$p = \frac{1}{2} \int_{-1-\varepsilon}^{-1+\varepsilon} j(c)dc + \frac{1}{2} \int_{1-\varepsilon}^{1+\varepsilon} j(c)dc.$$

Now we could use delta functions at 1 and -1 to define $j(\cdot)$, but this would just be another way of getting an optimum prediction algorithm. More usually, $j(\cdot)$ would be characterized by something that at least would be unimodal and symmetric, in which case we would have

$$p = \int_{1-\varepsilon}^{1+\varepsilon} j(c)dc \leq j(1-\varepsilon)2\varepsilon$$

Now

$$\int_{o}^{1-\varepsilon} j(c)dc \leq \frac{1}{2}$$

and

$$\int_{o}^{1-\varepsilon} j(c)dc \geq (1-\varepsilon)j(1-\varepsilon)$$

so

$$p \leq \frac{2\varepsilon}{2(1-\varepsilon)} = \frac{\varepsilon}{1-\varepsilon}$$

Therefore p is small when ε is small rather than $\frac{1}{2}$ for any value of ε as is the case for the optimum predictor.

Nevertheless, dither helped the mean estimator since we would have

$$p = 0$$

for $f = 0$ and $j(\cdot) = \delta(\cdot)$.

APPENDIX B

In this section we shall derive equation 6.3. For convenience we shall just consider the two-dimensional probability distribution function g(x) where we have suppressed the dependence on y. We shall also consider for convenience that g(x) has zero mean.

Then

$$\sigma_r^2 = \int (x_1^2 + x_2^2) g(x) dx_1 dx_2.$$

We can change to polar coordinates and have

$$\sigma_r^2 = \int r^2 g(r,\theta) r dr d\theta$$

$$1 = \int g(r,\theta) r dr d\theta,$$

and we can define

$$g^*(r) = \int_0^{2\pi} g(r,\theta) d\theta.$$

With this definition we have

$$\int_0^\infty g^*(r) r dr = 1$$

and

$$\sigma_r^2 = \int_0^\infty r^3 g^*(r) dr$$

Now let

$$\beta = \sup g(x),$$

and let

$$\gamma^2 = \frac{1}{\pi\beta}.$$

Then

$$0 \le g^*(r) \le 2\pi\beta.$$

We can write

$$\sigma_r^2 = \int_0^\gamma r^3 g^*(r)dr + \int_\gamma^\infty r^3 g^*(r)dr$$

$$\geq \int_0^\gamma r^3 g^*(r)dr + \gamma^2 \int_\gamma^\infty rg^*(r)dr$$

$$= \int_0^\gamma r^3 g^*(r)dr + \gamma^2 \left\{ 1 - \int_0^\gamma rg^*(r)dr \right\}$$

$$= \int_0^\gamma r^3 g^*(r)dr + \gamma^2 \left\{ \int_0^\gamma \frac{2}{\gamma^2} rdr - \int_0^\gamma rg^*(r)dr \right\}$$

$$= \int_0^\gamma r^3 g^*(r)dr + \gamma^2 \int_0^\gamma [2\pi\beta - g^*(r)]rdr$$

$$\geq \int_0^\gamma r^3 g^*(r)dr + \int_0^\gamma [2\pi\beta - g^*(r)]r^3 dr$$

This last step uses the fact that

$$[2\pi\beta - g^*(r)] \geq 0.$$

Finally we have

$$\sigma_r^2 \geq \int_0^\gamma 2\pi\beta r^3 dr = \frac{\pi\beta\gamma^4}{2} = \frac{1}{2\pi\beta}$$

Now we turn our attention to

$$\hat{p} = A \int \hat{g}(y)dm(y)$$

and

$$\sigma_r^2 = \int \sigma_r^2(y)dm(y)$$

where

$$\sigma_r^2(y) = \int |x - \mu(y)|^2 g(x,y)dx$$

and

$$\hat{g}(y) = \sup_x g(x,y)$$

From what we just proved we can write

$$\sigma_r^{\ 2} \geq \int \frac{1}{2\pi\hat{g}(y)}\, dm(y).$$

Now we have

$$\sigma_r^{\ 2}\ \hat{p} \geq \int \left\{ \sqrt{\frac{1}{2\pi\hat{g}(y)}} \right\}^2 dm(y) \int \left\{ \sqrt{A g(\hat{y})} \right\}^2 dm(y)$$

which gives

$$\geq \int \sqrt{\frac{1}{2\pi\hat{g}(y)}}\ \sqrt{A\hat{g}(y)}\ \ dm(y)$$

$$= \frac{A}{2\pi}$$

by a version of Schwartz's lemma and the fact that m(y) is a probability measure. This, of course, gives the final result

$$\hat{p} \geq \frac{A}{2\pi\sigma_r^{\ 2}}$$

that appears in equation 6.3.

If we had been considering a one-dimensional situation, we would have had

$$\sup g(x) \geq \frac{1}{2\sqrt{3}\ \sigma_x}$$

and

$$\hat{p} \geq \frac{L}{2\sqrt{3}\ \sigma_x}$$

where L is a "vulnerable length." We would also have as the analogues of equations 6.7 and 6.8

$$\frac{1}{\sqrt{2\pi}\ \sigma_{xw}}\, e^{-\frac{(\Delta x)^2}{2\sigma_{xw}^2}}$$

and

$$\hat{p} = \frac{L}{\sqrt{2\pi}\ \sigma_{xw}}$$

APPENDIX C

The argument in Reference 4 that is referred to in Section 6 of this paper goes as follows.

Consider the one-dimensional problem. The optimal (r.m.s.) linear predictor is given by

$$x(t+T) \cong \sum_{m=0}^{M-1} x^{(m)}(t)\,\frac{T^m}{m!} + \int_0^\infty x^{(M)}(t-\xi)f(\xi)d\xi \qquad \text{C-1}$$

where

$$E\left[x^{(M)}(t_1)x^{(M)}(t_2)\right] = \phi_M(t_1 - t_2) \qquad \text{C-2}$$

and

$$\phi_M(0) = C_M{}^2 < \infty\,. \qquad \text{C-3}$$

Now we can write

$$\phi_M(\tau) = \int_0^\infty \psi_M(t)\psi_M(t+\tau) \qquad \text{C-4}$$

$$\psi_M(t) = 0 \quad \text{for} \quad t < 0 \qquad \text{C-5}$$

$$C_M{}^2 = \int_0^\infty \psi_M(t)^2 dt. \qquad \text{C-6}$$

If we minimize the expected value of the square error of equation C-1 by picking $f(\cdot)$, we have for the residual error

$$\varepsilon^2 = \int_0^T ds \left[\int_0^s \frac{(s-r)^{M-1}}{(M-1)!}\,\psi_M(r)dr\right]^2 \qquad \text{C-7}$$

The pilot of the target can maximize ε^2 by choosing optimal statistics represented by ψ_M subject to C-6. This maximization gives

$$\varepsilon^2 = k_M C_M{}^2 \qquad \text{C-8}$$

where k_M is the largest eigenvalue of the differential equation

$$\psi_M^{\;L}(T) = 0 \qquad L = 0 \text{ to } M - 1 \tag{C-9}$$

$$\psi_M^{\;L+M}(0) = 0 \qquad L = 0 \text{ to } M - 1 \tag{C-10}$$

$$k_M \psi_M^{\;(2M)}(t) = (-1)^M \psi_M(t). \tag{C-11}$$

Likewise in two dimensions, if

$$C_M^{\;2} = E\left\{\left[x_1^{(M)}(t)^{\;2} + x_2^{(M)}(t)\right]^2\right\} \tag{C-12}$$

then

$$\sigma^2_{rwM} = k_M \, C_M^{\;2}. \tag{C-13}$$

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